QUANTUM PHYSICS I - Feb. 1, 2018

Write your name and student number on **all** answer sheets. There are four problems in this exam. You can earn 90 points in total, with exam grade equal to 1 + points/10.

PROBLEM 1: WAVEFUNCTION (10,10 and 5 points)

a) Prove that the normalization of the wavefunction

$$\int_{-\infty}^{+\infty} |\psi(x,t)|^2 dx = 1\,, \tag{1}$$

is time-independent (in other words, if you normalize your wavefunction once it will remain normalized).

b) Prove that the first derivative of the wavefunction, $d\psi/dx$, can have a discontinuity given by

$$\Delta\left(\frac{d\psi}{dx}\right) = \frac{2m}{\hbar^2}\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} V(x)\psi(x)dx\,. \tag{2}$$

This represents the discontinuity of the first derivative at $x = 0$.

c) What does this mean for a wavefunction when the potential energy is 1) finite at $x = 0$, or 2) has a delta function at $x = 0$?

PROBLEM 2: DELTA-FUNCTION POTENTIALS (10, 10, 5 points)

Consider a one-dimensional wavefunction $\psi(x)$ in the presence of delta-function potentials.

a) In the case of a single delta-function potential energy with (α is positive)

$$V = -\alpha\delta(x)\,, \tag{3}$$

derive the bound state solution for $\psi(x)$ (up to an arbitrary normalization). What is its energy?

b) In the case of a double delta-function potential energy with

$$V = -\alpha\delta(x + x_0) - \alpha\delta(x - x_0)\,, \tag{4}$$

sketch the wavefunction of the ground state and the first excited state.

c) Suppose one would put two identical particles in the system with a double delta-function, briefly describe what the wavefunction of the ground state would look like. Give a separate description of the case of 1) bosons and 2) fermions.

PROBLEM 3: HERMITIAN OPERATORS (all 5 points)

Consider Hermitian operators and their properties, e.g. eigenvectors, -functions and eigenvalues, in finite - and infinite-dimensional vector spaces.

a) Briefly explain one difference between the properties of Hermitian operators of finite-dimensional vector spaces and infinite-dimensional vector spaces with a discrete spectrum.

b) Briefly explain one difference between the properties of Hermitian operators of infinite-dimensional vector spaces with a discrete spectrum and with a continuous spectrum.

PROBLEM 4: ANGULAR MOMENTUM (10, 5, 5 and 10 points)

Angular momentum in three spatial dimensions is given by the combination

$$\vec{L} = \vec{x} \times \vec{p}. \tag{5}$$

a) Construct ladder operators $L_\pm$ by taking linear combinations of these components. What are the commutation relations between the ladder operators themselves, and with L_z?

b) How many commuting operators can one construct out of angular momentum? Which are they?

c) Consider a spherically symmetric potential with a generic r-dependence. Which quantum numbers label the different eigenstates, and what is the degeneracy of each eigenstate (i.e. the number of inequivalent states with the same energy)?

d) Describe in words what happens for the special case of a spherically symmetric potential of the Coulomb type, as in hydrogen. Which quantum numbers label the different eigenstates, and what is the degeneracy of each eigenstate (i.e. the number of inequivalent states with the same energy)?